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DENSELY DEFINED SELECTIONS OF MULTIVALUED MAPPINGS

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ABSTRACT. Rather general suficient conditions are found for a given multivalued mapping $F: X \to Y$ to possess an upper semicontinuous and compact-valued selection G which is defined on a dense G_δ-subset of the domain of F. The case when the selection G is single-valued (and continuous) is also investigated. The results are applied to prove some known as well as new results concerning generic differentiability of convex functions, Lavrentieff type theorem, generic well-posedness of optimization problems and generic non-multivaluedness of metric projections and antiprojections.

1. INTRODUCTION

Let $F: X \to Y$ be a multivalued mapping with nonempty images acting from the topological space X into the topological space Y (i.e., for each $x \in X$ the value $F(x)$ is a nonempty subset of Y). We give here sufficient conditions for the existence of an upper semicontinuous and nonempty compact-valued (u.s.c.o.) mapping $G: X_1 \to Y$, where X_1 is a dense G_δ-subset of X and G is a selection of F. In contrast to what is widely accepted under "*a selection of F on X_1*" we understand here a set-valued mapping $G: X_1 \to Y$ such that $\varnothing \neq G(x) \subset F(x)$ for every $x \in X_1$. In some particular cases the selection G coincides with the restriction of F on X_1. If the range space Y is completely metrizable, then G can be considered to be single-valued.

More precisely, we call F *lower demicontinuous* (l.d.c.) in X if for every open V in Y, the interior of the closure of the set $F^{-1}(V)\colon = \{x \in X : F(x) \cap V \neq \varnothing\}$ is dense in the closure of $F^{-1}(V)$, i.e., $\operatorname{Int}\operatorname{Cl}(F^{-1}(V))$ is dense in $\operatorname{Cl}(F^{-1}(V))$. The consept of l.d.s. mappings will be discussed in more detail in §4. Several results are proved in §4 asserting that a given l.d.c. mapping F admits an u.s.c.o. selection defined on a dense subset of X. A typical theorem (see Theorem 4.6 for full generality) reads as follows:

Let $F: X \to Y$ be a l.d.c. mapping with closed graph, where X is a Baire space and Y is Čech complete. Then there exist a dense G_δ-subset X_1 of X and an u.s.c.o. mapping $G: X_1 \to Y$ which is a selection of F. If, in addition, Y is completely metrizable, then G can be considered to be single-valued (see

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Theorem 4.7). In some cases the mapping G can be constructed in such a way that its values belong to a subset Y_1 of Y which is specified in advance.

Theorem 4.6 should be compared to the following theorem of E. Michael [M2] (although, the two results are different in nature): if the space X is paracompact, Y is a complete metric space and F is lower semicontinuous and closed-valued, then an u.s.c.o. selection H of F exists which is defined on all of the space X.

Theorems 4.8 and 4.9 generalize some of the results in the paper of E. Michael [M4]. Our example 4.15 gives a negative answer to Question 7.4 from the same paper of Michael [M4].

In §5 we describe situations in which the mapping F itself (not merely a selection of it) is u.s.c.o. at the points of a dense G_δ-subset of X. A partial case of Theorem 5.2 says that, if $F: X \to Y$ has a closed graph, X is a Baire space, Y is Čech complete and for every open $V \subset Y$ the interior of the set $\{x \in X: F(x) \subset V\}$ is dense in $F^{-1}(V)$, then there exists a dense G_δ-subset X_1 of X at the points of which F is u.s.c.o. If, in addition, Y is completely metrizable then F is single-valued at the points of X_1 (see Theorem 5.3). Theorem 5.3 is used to derive known results about generic nonmultivaluedness of metric projections and antiprojections (Theorem 6.10).

Theorem 5.4 is applied in §6 to prove a new version of the classical Lavrentieff theorem concerning the extension of a densely defined homeomorphism to a subset containing a dense G_δ-subset of the domain space (Theorem 6.4). As shown in Theorem 6.5 the selection theorems could be used to prove that a given convex and continuous function is differentiable at the points of some dense G_δ-subset of its domain.

Finally, Theorems 6.1 and 6.2 show that the main results from our previous papers [ČK1, ČK2, ČKR1, and ČKR2] are, in essence, easily derived from the results in §4.

The general construction used in the proofs of our results is described in Theorems 3.1 and 3.3.

2. Preliminaries

We recall briefly some notions and continuity properties of multivalued mappings. Let $F: X \to Y$, where X and Y are Hausdorff topological spaces, be such a mapping. By $\mathrm{Dom}(F)$, as usual, we denote the *domain* of F, i.e., the set $\{x \in X: F(x) \neq \varnothing\}$. If A and B are subsets of X and Y respectively, then by $F(A)$ we designate the set $\bigcup\{F(x): x \in A\}$ and by $F^{-1}(B)$ and $F^{\#}(B)$ the sets $\{x \in X: F(x) \cap B \neq \varnothing\}$ and $\{x \in X: F(x) \subset B\}$ correspondingly. In particular $F^{\#}(B)$ contains all points $x \in X$ for which $F(x) = \varnothing$. The set $\mathrm{Gr}(F) := \{(x, y) \in X \times Y: y \in F(x)\}$ denotes the *graph* of F. We will say that F has a closed graph if $\mathrm{Gr}(F)$ is a closed subset of the product topology in $X \times Y$. F is said to be *upper* (resp. *lower*) *semicontinuous* at a point $x_0 \in X$ if for every open $V \subset Y$ such that $F(x_0) \subset V$ (resp. $F(x_0) \cap V \neq \varnothing$) the set $F^{\#}(V)$ (resp. the set $F^{-1}(V)$) contains an open neighborhood of x_0. In this case we will write that F is u.s.c. (resp. l.s.c.) at x_0. F is u.s.c. (resp. l.s.c.) in X if it is u.s.c. (resp. l.s.c.) at any point of X. F is called u.s.c.o. if it is u.s.c. and compact-valued. Every u.s.c.o. mapping F has a closed graph.

Conversely, if the range space Y is compact and F has a closed graph then F is u.s.c.o.

An u.s.c.o. F is *minimal* if its graph does not contain properly the graph of any other u.s.c.o. mapping with the same domain. It can be seen (by the Kuratowski-Zorn lemma) that for every u.s.c.o. mapping $F: X \to Y$ there exists a minimal u.s.c.o. $G: X \to Y$ which is contained in F and has the same domain. The next statement is well-known and summarizes what we will need about minimal u.s.c.o. mappings.

Proposition 2.1. *The following are equivalent for an u.s.c.o.* $F: X \to Y$:

(a) F *is minimal*;

(b) *for every open* U *in* X *and closed* B *in* Y *from* $F(x) \cap B \neq \varnothing$ *for every* $x \in U \cap \mathrm{Dom}(F)$, *it follows that* $F(U) \subset B$;

(c) *if* U *and* V *are open subsets of* X *and* Y *such that* $(U \times V) \cap \mathrm{Gr}(F) \neq \varnothing$ *then there is a nonempty open* $U' \subset U$ *with* $U' \cap \mathrm{Dom}(F) \neq \varnothing$ *and* $F(U') \subset V$.

In the sequel only Hausdorff topological spaces will be considered, although some of the results are true for spaces which are not necessarily Hausdorff. For a given subset A of the topological space Z we denote by $\mathrm{Cl}_Z(A)$ and $\mathrm{Int}_Z(A)$ the closure and the interior of A in Z. If there is no danger of ambiguity we will omit the subscript Z. The completely regular space Z is said to be *Čech complete* if it lies as a G_δ-subset in its Stone-Čech compactification βZ (or in any other compactification of Z).

Slightly modifying an idea from [CČN], under a *sieve* $\mathbf{s}$ in the space Z we mean a sequence of indexed families of nonempty subsets $\{V_\alpha: \alpha \in A_n\}$, $n \geq 0$, of Z and a sequence of single-valued mappings $\pi_n: A_{n+1} \to A_n$, such that $V_\alpha = Z$ for $\alpha \in A_0$ and for every $\alpha \in A_n$, $n \geq 0$, $V_\alpha \supset \bigcup\{V_\beta: \beta \in \pi_n^{-1}(\alpha)\}$. Observe that we do not require $\{V_\beta: \beta \in \pi_n^{-1}(\alpha)\}$ to be a cover for V_α, $\alpha \in A_n$, $n \geq 0$ (and hence $\{V_\alpha: \alpha \in A_n\}$ need not to be a cover for Z). We remark also that the sets V_α, $\alpha \in A_n$, $n \geq 0$, need not be open. A sequence $\{\alpha_n\}$, $\alpha_n \in A_n$, for this sieve is called a *π-chain* if $\pi_n(\alpha_{n+1}) = \alpha_n$ for every $n \geq 0$. By $K(\mathbf{s})$ we denote the *kernel* of $\mathbf{s}$, i.e. the set $\{x \in Z: x \in \bigcap_{n=0}^\infty V_{\alpha_n}$ for some π-chain $\{\alpha_n\}\}$.

The sieve $\mathbf{s}$ is called *complete* if for each π-chain $\{\alpha_n\}$ the set $\bigcap_{n=0}^\infty V_{\alpha_n}$ is nonempty and compact and for every open U in Z such that $\bigcap_{n=0}^\infty V_{\alpha_n} \subset U$ there exists some n with $V_{\alpha_n} \subset U$. The sieve $\mathbf{s}$ is *open* if every V_α is open.

3. Main construction and basic results

This section is devoted to the existence of an u.s.c.o. selection of a given multivalued mapping $F: X \to Y$ which is defined on a residual subset of the domain space X. Recall that a subset A of the topological space X is *residual* in X if its complement $X \backslash A$ is of first Baire category in X, i.e., $X \backslash A = \bigcup_{n=1}^\infty H_n$, where $\mathrm{Int}\,\mathrm{Cl}(H_n) = \varnothing$. Every dense G_δ-subset of the space X is residual in X. The space X is said to be a *Baire space* if the intersection of every countable family of open and dense subsets of X is dense in X. By a *selection of* F on the subset A of $\mathrm{Dom}(F)$, we mean a multivalued mapping $G: A \to Y$ such that $\varnothing \neq G(x) \subset F(x)$ for every $x \in A$. We start with the main result in this paper.

Theorem 3.1. *Let $F\colon X \to Y$ be a mapping with closed graph and such that* $\mathrm{Dom}(F)$ *is dense in* X. *Let* X *be a Baire space and* Y *admit a complete sieve* $\mathbf{s} = (\{V_\alpha\colon \alpha \in A_n\}, \{\pi_n\})$ *such that* $\bigcup\{\mathrm{Int}\,\mathrm{Cl}\,F^{-1}(V_\beta)\colon \beta \in \pi_n^{-1}(\alpha)\}$ *is dense in* $\mathrm{Int}\,\mathrm{Cl}\,F^{-1}(V_\alpha)$ *for every* $\alpha \in A_n$ *and* $n \geq 0$. *Then there exist a dense* G_δ*-subset* X_1 *of* X *and an u.s.c.o. mapping* $G\colon X_1 \to K(\mathbf{s})$ *from* X_1 *into the kernel of the sieve* $\mathbf{s}$, *such that*:

(a) $X_1 \subset \mathrm{Dom}(F)$;

(b) G *is a selection of* F *on the set* X_1.

Proof. Let us consider the complete sieve $\mathbf{s} = (\{V_\alpha\colon \alpha \in A_n\}, \{\pi_n\})$ in Y. Give the sets A_n well-ordering in such a way that the mappings π_n are monotone, i.e., if $\beta, \gamma \in A_{n+1}$ and $\gamma < \beta$ then $\pi_n(\gamma) \leq \pi_n(\beta)$. Put now for every $\alpha \in A_n$, $n \geq 0$, $W_\alpha := \mathrm{Int}\,\mathrm{Cl}\,F^{-1}(V_\alpha)$, $U_\alpha := W_\alpha \setminus \mathrm{Cl}(\bigcup\{W_\beta\colon \beta \in A_n, \beta < \alpha\})$ and let $B_n\colon = \{\alpha \in A_n\colon U_\alpha \neq \varnothing\}$. Observe that, for every $n \geq 0$ the family of open sets $\{U_\alpha\colon \alpha \in B_n\}$ is pairwise disjoint by construction. Moreover, for every $n \geq 0$ and $\alpha \in B_n$, by monotonicity of π_n and the fact that $\bigcup\{\mathrm{Int}\,\mathrm{Cl}\,F^{-1}(V_\beta)\colon \beta \in \pi_n^{-1}(\alpha)\} = \bigcup\{W_\beta\colon \beta \in \pi_n^{-1}(\alpha)\}$ is dense in $\mathrm{Int}\,\mathrm{Cl}\,F^{-1}(V_\alpha) = W_\alpha$ for every $\alpha \in A_n$, we get $U_\beta \subset U_\alpha$ for every $\beta \in \pi_n^{-1}(\alpha)$. Let

$$U_n\colon = \bigcup\{U_\alpha\colon \alpha \in B_n\}, \qquad n \geq 0.$$

Each U_n is open in X. We claim that U_n are also dense in X.

First, observe that $\bigcup\{W_\alpha\colon \alpha \in A_n\}$ is dense in X for every *n*. For $n = 0$ this is obviously true since for $\alpha \in A_0$, $F^{-1}(V_\alpha) = F^{-1}(Y) = \mathrm{Dom}(F)$ and hence $W_\alpha = X$. For $n > 0$ the density follows by induction from the assumption that $\bigcup\{W_\beta\colon \beta \in \pi_n^{-1}(\alpha)\}$ is dense in W_α for every $\alpha \in A_n$.

To prove the denseness of U_n, $n \geq 0$, take a nonempty open set $U \subset X$. Since $\bigcup\{W_\alpha\colon \alpha \in A_n\}$ is dense in X there is at least one $\alpha \in A_n$ such that $U \cap W_\alpha \neq \varnothing$. Take the minimal such α. This implies $U \cap W_{\alpha'} = \varnothing$ for every $\alpha' < \alpha$ and $\alpha' \in A_n$. Hence $U \cap U_\alpha \neq \varnothing$.

Let $X_1 := \bigcap_{n=0}^\infty U_n$. Since X is a Baire space the set X_1 is dense G_δ in X. Each $x \in X_1$ belongs to just one set from the disjoint family $\{U_\alpha\colon \alpha \in B_n\}$. Therefore, it determines in a unique way a π-chain $\{\alpha_n(x)\}$, $\alpha_n(x) \in B_n$, such that $x \in \bigcap_{n=0}^\infty U_{\alpha_n(x)}$

Consider the multivalued mapping $\Phi\colon X_1 \to K(\mathbf{s})$ determined by

$$\Phi(x)\colon = \bigcap_{n=0}^{\infty} V_{\alpha_n(x)}, \qquad x \in X_1.$$

The mapping Φ is compact-valued and has nonempty images, i.e., $\mathrm{Dom}(\Phi) = X_1$. Moreover, Φ is u.s.c. Indeed, let V be an open subset of Y such that $\Phi(x) \subset V$ for some $x \in X_1$. Since the sieve $\mathbf{s}$ is complete we get some $\alpha_n(x) \in B_n$ such that $V_{\alpha_n(x)} \subset V$. Then it is easy to check that $\Phi(x') \subset V_{\alpha_n(x)}$ for every $x' \in U_{\alpha_n(x)} \cap X_1$. Hence Φ is u.s.c.o. in X_1.

Finally, let us define the mapping $G\colon X_1 \to K(\mathbf{s})$ by

$$G(x) := F(x) \cap \Phi(x), \qquad x \in X_1.$$

As an intersection of two mappings with graphs closed in $X_1 \times Y$, G also has a closed graph. We prove that G is nonempty-valued.

Take $x_0 \in X_1$ and suppose $G(x_0) = \varnothing$. This means that the nonempty compact (in $X \times Y$) set $\{(x_0, y): y \in \Phi(x_0)\}$ does not intersect the graph of F. Since $\mathrm{Gr}(F)$ is a closed subset of $X \times Y$, routine considerations show that there are open subsets U of X and V of Y such that $x_0 \in U$, $\Phi(x_0) \subset V$ and $(U \times V) \cap \mathrm{Gr}(F) = \varnothing$. That is, $F(x) \cap V = \varnothing$ for every $x \in U$. This is a contradiction since for some n, $V_{\alpha_n(x_0)} \subset V$ and, on the other hand, U has a nonempty intersection with $U_{\alpha_n(x_0)} \subset W_{\alpha_n(x_0)} = \operatorname{Int}\operatorname{Cl} F^{-1}(V_{\alpha_n(x_0)})$. Hence G is nonempty-valued. This implies $X_1 \subset \mathrm{Dom}(F)$.

Further, every nonempty-valued mapping with closed graph, which is contained in an u.s.c.o. mapping with the same domain, is u.s.c.o. itself. Hence G is an u.s.c.o. selection of F on X_1. The proof of the theorem is completed. □

Remark 3.2. The special case when $F = f^{-1}$ for some continuous and single-valued mapping $f: Y \to X$ and the sieve $\mathbf{s}$ is open deserves more attention. It was essentially considered by E. Michael in [M4] (see Theorem 4.12 below). It can be shown in this case, that, in addition to the conclusion of the above theorem, the set $C := G(X_1)$ is a G_δ-subset of Y. Indeed, with the notations in the proof above, let $H_n: = \bigcup\{f^{-1}(U_\alpha) \cap V_\alpha: \alpha \in B_n\}$. The continuity of f implies that every H_n is open in Y. Since for every π-chain $\{\alpha_n\}$, $\alpha_n \in B_n$, we have $F(\bigcap_{n=0}^\infty U_{\alpha_n}) = f^{-1}(\bigcap_{n=0}^\infty U_{\alpha_n}) = \bigcap_{n=0}^\infty f^{-1}(U_{\alpha_n})$ it is easily verified that $C = \bigcap_{n=0}^\infty H_n$.

We prove further, that sometimes the selection G from Theorem 3.1 coincides with the restriction of the mapping F on the set X_1. That is, the mapping F is itself u.s.c.o. in a dense G_δ-subset of its domain.

Theorem 3.3. *Let $F: X \to Y$ be a mapping with closed graph and* $\mathrm{Dom}(F)$ *be dense in X. Let X be a Baire space and Y possess a complete sieve $\mathbf{s} = (\{V_\alpha: \alpha \in A_n\}, \{\pi_n\})$ such that $\bigcup\{\operatorname{Int} F^\#(V_\beta): \beta \in \pi_n^{-1}(\alpha)\}$ is dense in* $\operatorname{Int} F^\#(V_\alpha)$ *for every $\alpha \in A_n$, $n \geq 0$. Then there exists a dense G_δ-subset X_1 of X such that $X_1 \subset$* $\mathrm{Dom}(F)$ *and the restriction $F|X_1$ of F on X_1 is an u.s.c.o. mapping from X_1 into $K(\mathbf{s})$. If, in addition, for every π-chain $\{\alpha_n\}$, $\alpha_n \in A_n$, $\bigcap_{n=0}^\infty V_{\alpha_n}$ is a one-point set then at the points of X_1, F is single-valued and u.s.c. as a mapping from X into Y.*

Proof. Let $\mathbf{s} = (\{V_\alpha: \alpha \in A_n\}, \{\pi_n\})$ be the complete sieve in Y. The proof of this theorem is similar to the proof of Theorem 3.1.

For every $\alpha \in A_n$, $n \geq 0$, instead of sets $\operatorname{Int}\operatorname{Cl}(F^{-1}(V_\alpha))$, we consider the sets $W_\alpha := \operatorname{Int} F^\#(V_\alpha)$. After well-ordering the sets A_n in a monotonic way with respect to the mappings π_n we put $U_\alpha = W_\alpha \setminus \mathrm{Cl}\,(\bigcup\{W_\beta: \beta < \alpha, \beta \in A_n\})$, $B_n := \{\alpha \in A_n: U_\alpha \neq \varnothing\}$, $U_n := \bigcup\{U_\alpha: \alpha \in B_n\}$, $n \geq 0$. In the same way as in Theorem 3.1 we prove that U_n are dense in X and hence $X_1 := \bigcap_{n=0}^\infty U_n$ is a dense G_δ-subset of X such that every $x \in X_1$ determines a unique π-chain $\{\alpha_n(x)\}$, $\alpha_n(x) \in B_n$, with $x \in \bigcap_{n=0}^\infty U_{\alpha_n(x)}$. We put $\Phi(x) := \bigcap_{n=0}^\infty V_{\alpha_n(x)}$. Φ is u.s.c.o. mapping from X_1 into $K(\mathbf{s})$. Observe that, by the construction $F(U_\alpha) \subset V_\alpha$ for every $\alpha \in B_n$, $n \geq 0$, and hence for each $x \in X_1$ $F(x) \subset \Phi(x)$. Therefore $G(x) := F(x) \cap \Phi(x) = F(x)$ for every $x \in X_1$.

As in the proof of Theorem 3.1 one proves (using that $\mathrm{Dom}(F)$ is dense in X and $\mathrm{Gr}(F)$ is closed in $X \times Y$) that F is nonempty-valued at the points of X_1. Hence $F|X_1: X_1 \to K(\mathbf{s})$ is an u.s.c.o. mapping with nonempty values.

Suppose, in addition, that for every π-chain $\{\alpha_n\}$ $\bigcap_{n=0}^{\infty} V_{\alpha_n}$ is a one-point set. Then Φ is single-valued and consequently for every $x \in X_1$ $F(x) = \Phi(x)$ is a singleton. Let $x_0 \in X_1$ and V be an open subset of Y such that $F(x_0) \in V$. Then for some n, $V_{\alpha_n(x_0)} \subset V$. Since $F(U_{\alpha_n(x_0)}) \subset V_{\alpha_n(x_0)}$ we get that F is u.s.c. at the points of X_1. □

Remark 3.4. In the proof above, the closedness of $\mathrm{Gr}(F)$ and the density of $\mathrm{Dom}(F)$ in X were used only to prove that the mapping F is nonempty-valued at the points of X_1. Therefore, the following modification of Theorem 3.3 is true: Let F act between the Baire space X and the topological space Y. Suppose Y admits a sieve $\mathbf{s}$ such that for every π-chain $\{\alpha_n\}$, $\alpha_n \in A_n$, $\bigcap_{n=0}^{\infty} V_{\alpha_n}$ is either empty or is a one-point set such that for every open V containing $\bigcap_{n=0}^{\infty} V_{\alpha_n}$ one has $V_{\alpha_n} \subset V$ for some n. Let $\bigcup\{\mathrm{Int}\, F^{\#}(V_\beta) : \beta \in \pi_n^{-1}(\alpha)\}$ be dense in $\mathrm{Int}\, F^{\#}(V_\alpha)$ for every $\alpha \in A_n$, $n \geq 0$. Then there is a dense G_δ-subset X_1 of X such that: for every $x \in X_1$, either $F(x) = \varnothing$ or F is single-valued and u.s.c. at x. In particular, if $\mathrm{Dom}(F) = X$ then F is single-valued and u.s.c. at any point of X_1.

4. Selection theorems for lower demicontinuous mappings

In this section we give some important cases when Theorem 3.1 can be applied. For this we need some more notions.

There are different continuity properties of multivalued mappings which are weaker than the lower semicontinuity (see for example [Chr, ChrK, DK, K1, and K2]). In this paper we shall be concerned with a further relaxation of the lower semicontinuity. We call a mapping $F: X \to Y$ *lower demicontinuous* (l.d.c.) in X if for every open V in Y the set $\mathrm{Int}\,\mathrm{Cl}(F^{-1}(V))$ is dense in $\mathrm{Cl}(F^{-1}(V))$.

Like in the case of single-valued maps (see [HS]), we call a multivalued mapping $F: X \to Y$ *demiopen* if $\mathrm{Int}_Y \mathrm{Cl}_Y(F(U))$ is dense in $\mathrm{Cl}_Y(F(U))$, provided U is open in X.

The next is elementary.

Proposition 4.1. *The following are equivalent for* $F: X \to Y$:

(a) F *is lower demicontinuous in* X;

(b) *for every couple* (U, V) *consisting of open subsets of* X *and* Y *respectively such that* $(U \times V) \cap \mathrm{Gr}(F) \neq \varnothing$ *one has* $\mathrm{Int}\,\mathrm{Cl}(U \cap F^{-1}(V)) \neq \varnothing$;

(c) *the restriction of the natural projection map* $\pi_X: X \times Y \to X$ *on* $\mathrm{Gr}(F)$ *is demiopen.*

In other words, F is lower demicontinuous in X iff for every open U in X and open V in Y such that $U \cap F^{-1}(V) \neq \varnothing$, the set $F^{-1}(V)$ is dense somewhere in U.

Obviously, lower semicontinuity of F in X implies its lower demicontinuity. Moreover, it is an immediate consequence from Propositions 2.1(c) and 4.1(b) that every nonempty-valued minimal u.s.c.o. F in X is lower demicontinuous in X.

Having in mind Proposition 4.1 we get immediately its dual.

Proposition 4.2. *The following are equivalent for* $F : X \to Y$:

(a) F *is demiopen*;

(b) F^{-1} *is lower demicontinuous*;

(c) *for every couple* (U, V) *of open subsets of* X *and* Y *such that* $(U \times V) \cap \mathrm{Gr}(F) \neq \varnothing$ *it is true that* $\mathrm{Int}_Y \mathrm{Cl}_Y(V \cap F(U)) \neq \varnothing$;

(d) *the restriction of the natural projection mapping* $\pi_Y: X \times Y \to Y$ *on* $\mathrm{Gr}(F)$ *is demiopen.*

Let Y' be a nonempty subset of Y. We say that the multivalued mapping $F: X \to Y$ *embraces* Y' if for every open W in Y containing Y' the set $\{(x, y) \in \mathrm{Gr}(F): y \in W\}$ is dense in $\mathrm{Gr}(F)$, i.e., $\pi_Y^{-1}(W) \cap \mathrm{Gr}(F)$ is dense in $\mathrm{Gr}(F)$.

Proposition 4.3. *The following are equivalent for* $F: X \to Y$, *provided* Y *is regular*:

(a) $F: X \to Y$ *embraces* $Y' \subset Y$;

(b) *for every open sets* V_1 *and* V_2 *in* Y *such that* $V_1 \subset V_2$ *and* $V_1 \cap Y' = V_2 \cap Y'$ *the set* $F^{-1}(V_1)$ *is dense in* $F^{-1}(V_2)$;

(c) *If* $\bigcup\{V_\alpha: \alpha \in A\} \subset V$, V_α, V *are open in* Y *and* $V \cap Y = \bigcup\{V_\alpha \cap Y': \alpha \in A\}$ *then the set* $\bigcup\{F^{-1}(V_\alpha): \alpha \in A\}$ *is dense in* $F^{-1}(V)$.

Proof. (b)$\Leftrightarrow$(c) is obvious. We prove (a)$\Rightarrow$(b). Take open subsets V_1 and V_2 of Y such that $V_1 \subset V_2$, $V_1 \cap Y' = V_2 \cap Y'$ and suppose that $F^{-1}(V_2) \backslash \mathrm{Cl}\, F^{-1}(V_1) \neq \varnothing$. Put $H := X \backslash \mathrm{Cl}\, F^{-1}(V_1)$ and take $x_0 \in H \cap F^{-1}(V_2)$. Hence there is $y_0 \in V_2 \cap F(x_0)$. Further, there exist disjoint open sets W_1 and W_2 of Y such that $y_0 \in W_1 \subset V_2$ and $Y \backslash V_2 \subset W_2$. Let $W := V_1 \cup W_2$. Then, by construction $Y' \subset W$. Consider the set $B := \{(x, y) \in \mathrm{Gr}(F): y \in W\}$. We have $(H \times W_1) \cap B = \varnothing$ while $(x_0, y_0) \in (H \times W_1) \cap \mathrm{Gr}(F)$. This is a contradiction.

Conversely, to prove (b)$\Rightarrow$(a), suppose that W is an open subset of Y such that $W \supset Y'$. Let U and V be nonempty open subsets of X and Y such that $(U \times V) \cap \mathrm{Gr}(F) \neq \varnothing$. Put $V_1 := W \cap V$. Obviously $V_1 \cap Y' = V \cap Y'$ and hence $F^{-1}(V_1)$ is dense in $F^{-1}(V)$. Since $U \cap F^{-1}(V) \neq \varnothing$ we get $U \cap F^{-1}(V_1) \neq \varnothing$. The proof is completed. □

Proposition 4.4. *Let the mapping* $F: X \to Y$ *embrace* $Y' \subset Y$ *and* Y *be regular. Then* $F(X) \subset \mathrm{Cl}_Y(Y')$.

The proof is straightforward and is omitted.

The following assertion (the proof of which is elementary) gives some sufficient conditions for a mapping $F: X \to Y$ to embrace a subset Y' of Y.

Proposition 4.5. *Each of the following conditions implies that* F *embraces* Y':

(a) $F(X) \subset Y'$;

(b) *the set* $\{(x, y) \in \mathrm{Gr}(F): y \in Y'\}$ *is dense in* $\mathrm{Gr}(F)$;

(c) Y' *is dense in* Y *and the mapping* F *is demiopen*;

(d) Y' *is dense in* Y *and* $F^{-1}: Y \to X$ *is lower demicontinuous.*

The topological space X is said to be *sieve complete* (see [ČČN and M3]) if it has a complete open sieve $\mathbf{s} = (\{V_\alpha: \alpha \in A_n\}, \{\pi_n\})$, $n \geq 0$, such that $V_\alpha = \bigcup\{V_\beta: \beta \in \pi_n^{-1}(\alpha)\}$ for every $\alpha \in A_n$, $n \geq 0$. Every Čech complete space is sieve complete. The converse is not true in general, but the notions coincide in the class of metrizable spaces (even in paracompact spaces) [ČČN, Remark 3.9d; M3, Theorem 3.2].

Now we are ready to prove

Theorem 4.6. *Let F be a lower demicontinuous mapping, acting from the Baire space X into the regular space Y which embraces a sieve complete subspace Y_1 of Y. Let F be with closed graph and* $\mathrm{Dom}(F)$ *be dense in X. Then there exist a dense G_δ-subset X_1 of X and an u.s.c.o. mapping $G: X_1 \to Y_1$ such that*:

(a) $X_1 \subset \mathrm{Dom}(F)$;

(b) *G is a selection of F on the set X_1.*

In particular, the requirements of this theorem are fulfilled if F is l.d.c., $\mathrm{Dom}(F) = X$, $\mathrm{Gr}(F)$ is closed, X is Baire, and Y is a Čech complete space.

Proof. Having in mind Proposition 4.4 we may assume that Y_1 is dense in Y.

Since Y_1 is sieve complete there exists a complete open sieve $\mathbf{s} = (\{V_\alpha : \alpha \in A_n\}, \{\pi_n\})$ in Y_1 such that $V_\alpha = \bigcup\{V_\beta : \beta \in \pi_n^{-1}(\alpha)\}$ for every $\alpha \in A_n$, $n \geq 0$. Let $\tilde{V}_\alpha$, $\alpha \in A_n$, $n \geq 0$, be open subsets of Y such that $\tilde{V}_\alpha = Y$ for $\alpha \in A_0$, and for every $\alpha \in A_n$, $n \geq 0$, we have:

(1) $V_\alpha = Y_1 \cap \tilde{V}_\alpha$ (and hence $\bigcup\{\tilde{V}_\beta \cap Y_1 : \beta \in \pi_n^{-1}(\alpha)\} = \tilde{V}_\alpha \cap Y_1$);

(2) $\bigcup\{\tilde{V}_\beta : \beta \in \pi_n^{-1}(\alpha)\} \subset \tilde{V}_\alpha$;

(3) if $\{\alpha_n\}$ is a π-chain for $\mathbf{s}$ then $\bigcap_{n=0}^\infty \tilde{V}_{\alpha_n} = \bigcap_{n=0}^\infty V_{\alpha_n}$ and for every open V in Y such that $\bigcap_{n=0}^\infty \tilde{V}_{\alpha_n} \subset V$ one has $\tilde{V}_{\alpha_n} \subset V$ for some n.

The condition (3) is guaranteed by the regularity of Y and density of Y_1 in Y.

The conditions (2) and (3) above show that $\tilde{\mathbf{s}} = (\{\tilde{V}_\alpha : \alpha \in A_n\}, \{\pi_n\})$ is a complete open sieve in Y. So, with a slight abuse of notation we will designate $\tilde{\mathbf{s}}$ by $\mathbf{s}$ and $\tilde{V}_\alpha$ by V_α.

Since in our case $K(\mathbf{s}) \subset Y_1$, to get the conclusion of the theorem as a consequence of Theorem 3.1 we have only to show that $\bigcup\{\mathrm{Int}\,\mathrm{Cl}\,F^{-1}(V_\beta) : \beta \in \pi_n^{-1}(\alpha)\}$ is dense in $\mathrm{Int}\,\mathrm{Cl}\,F^{-1}(V_\alpha)$ for every $\alpha \in A_n$, $n \geq 0$. But this is a straightforward consequence from (1) above, Proposition 4.3(c) (the fact that F embraces Y_1) and lower demicontinuity of F. $\square$

Theorem 4.7. *Let F be a lower demicontinuous mapping acting from the Baire space X into the regular space Y. Let* $\mathrm{Dom}(F)$ *be dense in X, F be with closed graph and F embrace a completely metrizable subspace Y_1 of Y. Then there exist a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f: X_1 \to Y_1$ such that $X_1 \subset$* $\mathrm{Dom}(F)$ *and f is a selection of F on X_1.*

Proof. Let ρ be the complete metric inducing the topology in Y_1. In Y_1 we consider the following complete open sieve $\mathbf{s}$: for $n = 0$, the index set A_0 consists of only one element and $V_\alpha = Y_1$, $\alpha \in A_0$. If the sieve is constructed up to some n, $n \geq 0$, the next step is the following: for every $\alpha \in A_n$ consider the family $\{V \subset Y_1 : V$ is open in Y_1, $\mathrm{Cl}_{Y_1}(V) \subset V_\alpha$ and $\rho\text{-}\mathrm{diam}(V) < 1/(n+1)\}$. Index that family by some set $A_{n+1}(\alpha)$ and put $A_{n+1} := \bigcup\{A_{n+1}(\alpha) : \alpha \in A_n\}$ and $\pi_n^{-1}(\alpha) := A_{n+1}(\alpha)$. In this way the sieve $\mathbf{s}$ is defined and it is easily seen that $\mathbf{s}$ is a complete open sieve in Y_1 for which, the set $\bigcap_{n=0}^\infty V_{\alpha_n}$ is a one-point set whenever $\{\alpha_n\}$, $\alpha_n \in A_n$, is a π-chain. To finish, we proceed as in the proof above. $\square$

Call the space X *almost complete* if it contains a dense sieve complete subspace. The class of such spaces was originally introduced in 1960 by Frolík [Fro] in a different way. If X is regular, Frolík's definition is equivalent to the above one [M4, Propositions 4.2 and 4.5]. This class was studied also by Aarts and Lutzer in [AL, §4], where they called these spaces almost Čech complete. It is true that (see [Fro]) a completely regular space X is almost complete iff it contains a dense and Čech complete subspace. For a detailed study of almost complete spaces (as well as spaces admitting similar completeness properties) we refer to the recent paper of E. Michael [M4]. Let us mention that every regular almost complete space is a Baire space (see, e.g., [M4, Proposition 4.5]).

We have the following important partial cases of the above theorems.

Theorem 4.8. *Let F be a l.d.c. and demiopen mapping from the Baire space X into the regular almost complete space Y. Let F be with closed graph and* $\mathrm{Dom}(F)$ *be dense in X. Then there exist a dense G_δ-subset X_1 of X and an u.s.c.o. mapping $G\colon X_1 \to Y$ such that:*

(a) $X_1 \subset \mathrm{Dom}(F)$;

(b) *G is a selection of F on X_1.*

Proof. Since F is demiopen it embraces every dense subspace of Y (Proposition 4.5(c)). Therefore, we can apply Theorem 4.6. □

Analogously, by Theorem 4.7 and Proposition 4.5(c) one obtains

Theorem 4.9. *In the assumptions of Theorem 4.8 suppose that Y contains a dense and completely metrizable subspace. Then the selection G can be considered to be single-valued.*

We give now some examples showing that the assumptions on F in Theorem 4.6 and Theorem 4.7 are essential.

Example 4.10. Let the segment $X := [0, 1]$ be endowed with the usual topology on the real line. Let further $B \subset [0, 1]$ be such that both B and $X\backslash B$ are dense Baire subspaces of X (e.g., let B be a Bernstein subset of $[0, 1]$, see [En, 4.5.5(b) and 5.5.4]). Put $Y := (B \times \{0\}) \cup ((X\backslash B) \times \{1\}) \cup (X \times (0, 1))$, where $(0, 1)$ is the open interval, and consider on Y the topology inherited by the product topology of $X \times X$. Let $F\colon X \to Y$ be defined by $F(x) = (x, 0)$ if $x \in B$ and $F(x) = (x, 1)$ if $x \in X\backslash B$. F has a closed graph because $B \times \{0\}$ and $(X\backslash B) \times \{1\}$ are closed subsets of Y. F is obviously lower demicontinuous. But there is no dense G_δ-subset of X on which F possesses a continuous selection. In this example F does not embrace any completely metrizable subspace of Y.

Example 4.11. Let X and B be as in Example 4.10. Define $F\colon X \to \{0, 1\}$ by $F(x) = 1$ if $x \in B$ and $F(x) = 0$ otherwise. F is l.d.c. but $\mathrm{Gr}(F)$ is not closed and F does not have a continuous selection on a dense G_δ-subset of X.

We give some further consequences of Theorem 4.6 and Theorem 4.7. Let us start with a result of Michael [M4].

Theorem 4.12 [M4, Theorem 7.2]. *Let $f\colon Y \to X$ be a continuous and demiopen single-valued mapping acting from the regular almost complete space Y*

into the space X such that $f(Y)$ is dense in X. Then there exist a G_δ-subset C of Y and a dense G_δ-subset D of X such that $f|C\colon C \to D$ is perfect and onto.

Proof. Let $F\colon = f^{-1}\colon X \to Y$. By the assumptions $\mathrm{Dom}(F) = f(Y)$ is dense in X and by [M4, Proposition 6.6], X is a Baire space. Since f is demiopen then F is l.d.c. On the other hand, the continuity of f implies that F is open and has closed graph. Therefore, by Theorem 4.8 there are a dense G_δ-subset D of X and an u.s.c.o. $G\colon D \to Y$ such that $D \subset \mathrm{Dom}(F) = f(Y)$ and G is a selection of F. Let $C := G(D)$. Since $F = f^{-1}$ and f is single-valued then $F(x_1) \cap F(x_2) = \varnothing$ for $x_1 \neq x_2$. Hence $G(x) = F(x) \cap C$ for every $x \in D$. This implies that $G = (f|C)^{-1}$ showing that $f|C\colon C \to D$ is perfect and onto. The rest follows by Remark 3.2. $\square$

Theorem 4.13 [M4, Theorem 7.3]. *If the space Y in Theorem* 4.12 *contains a dense and completely metrizable subspace then $f|C$ can be considered to be a homeomorphism.*

Proof. As above using Theorem 4.9. $\square$

Remark 4.14. It should be noted that Theorems 7.2 and 7.3 from the paper of E. Michael [M4] contain additional information about the sets C and D (special completeness properties). These completeness properties do not follow directly from our Theorems 4.8 and 4.9.

In connection with Theorem 4.13 the following questions were raised in [M4, Question 7.4]: Let f be an open and continuous single-valued mapping from a Čech complete space Y onto a regular (or even metrizable) space X. Must f map some nonempty subset C of Y homeomorphically onto a dense G_δ-subset of X? We give here a negative answer to one of these questions, namely, to the case when X is not metrizable.

Example 4.15. Let τ be a cardinal which is greater than or equal to the first uncountable cardinal. By a result of Pasinkov [Pas, Theorem 2], there are a compact space Y_τ of weight τ with $\dim Y_\tau = 1$, and a continuous and open mapping f which maps Y_τ onto $[0, 1]^\tau$ and such that $\dim(f^{-1}(x)) = 0$ for every $x \in [0, 1]^\tau$. Let us observe that if H is a nonempty G_δ-subset of $[0, 1]^\tau$ then H contains a homeomorphic image of $[0, 1]^\tau$. Suppose now that there is $C \subset Y_\tau$, $C \neq \varnothing$, such that $f(C) = H$, H is G_δ in $[0, 1]^\tau$, and $f|C$ is a homeomorphism. Then C contains a homeomorphic image of $[0, 1]^\tau$, so the same does Y_τ. But $\dim[0, 1]^\tau = \infty$ while $\dim Y_\tau = 1$. This is a contradiction.

At the end of this section we show that when the range space Y is second countable the conclusion of Theorem 4.7 can be obtained under weakened assumptions on the mapping F.

Theorem 4.16. *Let F be a mapping with closed graph acting from a Baire space X into the completely metrizable separable space Y. Let* $\mathrm{Dom}(F) = X$. *Then there exist a dense G_δ-subset X_1 of X and a single-valued continuous mapping $f\colon X_1 \to Y$ which is a selection of F on X_1.*

Proof. We follow a scheme from [K2]. Let $\{V_n\}$, $n \geq 1$, be a countable base for the topology in Y. Consider the sets $H_n = \mathrm{Cl}\, F^{-1}(V_n) \setminus \mathrm{Int}\,\mathrm{Cl}\, F^{-1}(V_n)$. The sets H_n are closed and nowhere dense in X and consequently the set

$X' := \bigcap_{n=1}^{\infty} X \backslash H_n$ is dense G_δ in X. Further, since $\{V_n\}$, $n \geq 1$, is a base in Y, it is routine matter to check that the restriction $F|X'$ of the mapping F on X' is l.d.c. To complete the proof, apply Theorem 4.7 for the mapping $F|X' \colon X' \to Y$. □

5. Generic continuity of multivalued mappings

In this section we prove several results in which a mapping $F \colon X \to Y$ is u.s.c.o. (or single-valued and u.s.c.) itself at the points of a residual subset of its domain. We start with a general result when the range space is metric.

A mapping $F \colon X \to Y$, where Y is a metric space with a metric ρ, is said to be *fragmented* by ρ (see [HJT, p. 217]) if for every $\varepsilon > 0$ and every nonempty open set U in X there exists a nonempty open $U' \subset U$ such that $\rho\text{-diam}(F(U')) < \varepsilon$.

Theorem 5.1. *Let $F \colon X \to Y$, where X is a Baire space and (Y, ρ) is a metric space, be fragmented by the metric ρ. Then there exists a dense G_δ-subset X_1 of X such that:*

(a) *for every $x \in X_1$ either $F(x) = \varnothing$ or F is single-valued and u.s.c. at x.*

(b) *if ρ is complete,* $\mathrm{Dom}(F)$ *is dense in X and F has a closed graph then $X_1 \subset \mathrm{Dom}(F)$ and at the points of X_1 F is single-valued and u.s.c.o.*

Proof. We construct a sieve $\mathbf{s}$ (not necessarily open) in Y as in the proof of Theorem 4.7. The only difference is the following: on the $(n+1)$th step for every $\alpha \in A_n$ we consider the family $\{V : \mathrm{Cl}_Y(V) \subset V_\alpha$ and $\rho\text{-diam}(V) < 1/(n+1)\}$, i.e., here we do not require that the sets V are open in Y. Observe that if the metric ρ is complete then the sieve $\mathbf{s}$ is complete too. For every π-chain $\{\alpha_n\}$, $\alpha_n \in A_n$, $\bigcap_{n=0}^{\infty} V_{\alpha_n}$ is either empty or is a one-point set. In the latter case for every open V in Y containing $\bigcap_{n=0}^{\infty} V_{\alpha_n}$ one has $V_{\alpha_n} \subset V$ for some n.

The so constructed sieve $\mathbf{s}$, the mapping F and the spaces X and Y satisfy the requirements of Theorem 3.3 (see also Remark 3.4). The proof is completed. □

Theorem 5.2. *Let F be a multivalued mapping acting from a Baire space X into the regular space Y such that* $\mathrm{Dom}(F)$ *is dense in X,* $\mathrm{Gr}(F)$ *is closed and F embraces some sieve complete subspace Y_1 of Y. Suppose that* $\mathrm{Int}\, F^{\#}(V)$ *is dense in $F^{-1}(V)$ for every open V in Y. Then there exists a dense G_δ-subset X_1 of X such that $X_1 \subset \mathrm{Dom}(F)$, F maps X_1 into Y_1, and at the points of X_1 F is u.s.c.o.*

Proof. As in the proof of Theorem 4.6 we may suppose that Y_1 is dense in Y and get a complete open sieve $\mathbf{s} = (\{V_\alpha \colon \alpha \in A_n\}, \{\pi_n\})$ in Y such that for every $\alpha \in A_n$, $n \geq 0$, $\bigcup\{V_\beta \cap Y_1 \colon \beta \in \pi_n^{-1}(\alpha)\} = V_\alpha \cap Y_1$ and $K(\mathbf{s}) \subset Y_1$. Since F embraces Y_1 we have, by Proposition 4.3(c), that for every $\alpha \in A_n$, $n \geq 0$, $\bigcup\{F^{-1}(V_\beta) \colon \beta \in \pi_n^{-1}(\alpha)\}$ is dense in $F^{-1}(V_\alpha)$. Now, since $\mathrm{Dom}(F)$ is dense in X and $\mathrm{Int}\, F^{\#}(V)$ is dense in $F^{-1}(V)$ for every open V in Y, it is easy to see that $\bigcup\{\mathrm{Int}\, F^{\#}(V_\beta) \colon \beta \in \pi_n^{-1}(\alpha)\}$ is dense in $\mathrm{Int}\, F^{\#}(V_\alpha)$. Hence, by Theorem 3.3 there is a dense G_δ-subset X_1 of X such that $X_1 \subset \mathrm{Dom}(F)$ and $F|X_1 \colon X_1 \to Y_1$ is u.s.c.o.

We prove now that at the points of X_1 the mapping $F \colon X \to Y$ is u.s.c. Take $x_0 \in X_1$ and let V be an open subset of Y such that $F(x_0) \subset V$. Since $F(x_0)$

is compact and Y is regular there is an open subset W of Y with $F(x_0) \subset W$ and $\mathrm{Cl}_Y(W) \subset V$. Having in mind that F is u.s.c. in X_1 there is an open subset U of X such that $F(x) \subset W$ for every $x \in U \cap X_1$. Suppose there is a point $x_1 \in U \cap \mathrm{Dom}(F)$ such that $F(x_1) \setminus \mathrm{Cl}_Y(W) \neq \varnothing$. Take y_0 from this set. Then there is an open in Y set V_0 such that $y_0 \in V_0$ and $V_0 \cap \mathrm{Cl}_Y(W) = \varnothing$. Since $\mathrm{Int}\, F^{\#}(V_0)$ is dense in $F^{-1}(V_0)$ there exists a nonempty open $U' \subset U$ such that $F(U') \subset V_0$. This is a contradiction since $U' \cap X_1$ is a nonempty subset of $U \cap X_1$. The proof is completed. □

In the same way one can prove

Theorem 5.3. *Under the assumptions of Theorem* 5.2 *suppose in addition that* Y_1 *is completely metrizable. Then* F *maps* X_1 *into* Y_1 *and* F *is single-valued and u.s.c. at the points of* X_1.

Let us mention that Remark 3.4 is valid also for Theorem 5.3.

We prove now another result of the above type. The proof is again based on the construction of inscribed families of disjoint open sets and it is a generalization of a construction from [KG].

Theorem 5.4. *Let* X *and* Y *be regular and contain completely metrizable dense subspaces* $X_0 \subset X$ *and* $Y_0 \subset Y$. *Let* $F\colon X \to Y$ *be a nonempty-valued mapping such that for every nonempty open* V *in* Y *and* U *in* X *the sets* $\mathrm{Int}_X F^{\#}(V)$ *and* $\mathrm{Int}_Y (F^{-1})^{\#}(U)$ *are nonempty. Then there are dense* G_δ*-subsets* X_1 *of* X *and* Y_1 *of* Y *such that*:

(a) $X_1 \subset X_0$ *and* $Y_1 \subset Y_0$;

(b) F *is single-valued and u.s.c. at the points of* X_1 *and* F^{-1} *is single-valued and u.s.c. at the points of* Y_1;

(c) *the restriction* $F|X_1$ *is a homeomorphism between* X_1 *and* Y_1.

Proof. It is clear that $\mathrm{Dom}(F^{-1}) = F(X)$ is dense in Y.

Let $X_0 \subset X$ and $Y_0 \subset Y$ be dense subsets of X and Y which are completely metrizable by some metrics ρ and d respectively. Take a maximal disjoint family $\Delta_1 := \{V : V \text{ is open in } Y,\ d\text{-diam}(V \cap Y_0) < 1\}$. Hence $V_1 := \bigcup\{V : V \in \Delta_1\}$ is dense in Y. In X consider the disjoint family $\Gamma_1' := \{\mathrm{Int}_X F^{\#}(V) \colon V \in \Delta_1\}$. We prove that $\bigcup\{U \colon U \in \Gamma_1'\}$ is dense in X. Let W be a nonempty open subset of X. Then $\mathrm{Int}_Y(F^{-1})^{\#}(W) \neq \varnothing$. Hence $H := V \cap \mathrm{Int}_Y(F^{-1})^{\#}(W)$ is a nonempty open subset of Y for some $V \in \Delta_1$. This implies $\varnothing \neq \mathrm{Int}_X F^{\#}(H) \subset \mathrm{Int}_X F^{\#}(V)$. Since F is nonempty-valued we have $\varnothing \neq \mathrm{Int}_X F^{\#}(H) \subset F^{-1}(H) \subset W$ and consequently $W \cap \mathrm{Int}_X F^{\#}(V) \neq \varnothing$.

Let us inscribe in Γ_1' a maximal disjoint family $\Gamma_1 := \{U \subset X \colon U \text{ is open in } X, \mathrm{Cl}_X(U) \subset U' \text{ for some } U' \in \Gamma_1' \text{ and } \rho\text{-diam}(U \cap X_0) < 1\}$. Since $\bigcup\{U' \colon U' \in \Gamma_1'\}$ is dense in X the same is true for $U_1 := \bigcup\{U \colon U \in \Gamma_1\}$.

Now, consider $\Delta_1' := \{\mathrm{Int}_Y(F^{-1})^{\#}(U) \colon U \in \Gamma_1\}$. This family is again disjoint and since the union of the elements of Γ_1 is dense in X one can prove, as above (using the fact that $\mathrm{Dom}(F^{-1})$ is dense in Y), that $\bigcup\{H \colon H \in \Delta_1'\}$ is dense in Y. Obviously for every $H \in \Delta_1'$, there exists some $V \in \Delta_1$ such that $H \cap V \neq \varnothing$. Since Y is regular, this allows to inscribe simultaneously in Δ_1' and Δ_1 a maximal disjoint family $\Delta_2 \colon = \{V \colon V \text{ is open in } Y, d\text{-diam}(V \cap Y_0) < 1/2,\ V \subset H \text{ for some } H \in \Delta_1' \text{ and } \mathrm{Cl}_Y(V) \subset V' \text{ for some } V' \in \Delta_1\}$. By the maximality of Δ_2 and the fact that $\bigcup\{V \colon V \in \Delta_1\}$

and $\bigcup\{V: V \in \Delta_1'\}$ are dense in Y, it follows that $V_2: = \bigcup\{V: V \in \Delta_2\}$ is dense in Y. Proceeding in this way we obtain sequences of disjoint families $\{\Gamma_n\}$ and $\{\Delta_n\}$ of open sets in X and Y respectively such that:

(i) $U_n: = \bigcup\{U: U \in \Gamma_n\}$ and $V_n: = \bigcup\{V: V \in \Delta_n\}$ are dense in X and Y respectively for every n;

(ii) for every $U \in \Gamma_{n+1}$ and $V \in \Delta_{n+1}$ there are $U' \in \Gamma_n$ and $V' \in \Delta_n$ such that $\mathrm{Cl}_X(U) \subset U'$ and $\mathrm{Cl}_Y(V) \subset V'$;

(iii) for every $U \in \Gamma_n$ there is $V \in \Delta_n$ such that $F(U) \subset V$ and for every $V \in \Delta_{n+1}$ there is $U \in \Gamma_n$ such that $F^{-1}(V) \subset U$;

(iv) for every $U \in \Gamma_n$, $V \in \Delta_n$ ρ-diam$(U \cap X_0) < 1/n$ and d-diam$(V \cap Y_0) < 1/n$.

Let $X_1 := \bigcap_{n=1}^{\infty} U_n$ and $Y_1 := \bigcap_{n=1}^{\infty} V_n$. Obviously X_1 and Y_1 are dense G_δ-subsets of X and Y. Moreover, (iv) and the regularity of X and Y imply that $X_1 \subset X_0$ and $Y_1 \subset Y_0$.

Each $x \in X_1$ defines in a unique way sequences $\{U_n(x)\}$ and $\{V_n(x)\}$ such that $U_n(x) \in \Gamma_n$, $V_n(x) \in \Delta_n$, $x \in \bigcap_{n=1}^{\infty} U_n(x)$, $\mathrm{Cl}_X(U_{n+1}(x)) \subset U_n(x)$, $\mathrm{Cl}_Y(V_{n+1}(x)) \subset V_n(x)$, $F(U_n(x)) \subset V_n(x)$, and $F^{-1}(V_{n+1}(x)) \subset U_n(x)$. Moreover, since ρ-diam$(U_n(x) \bigcap X_0) < 1/n$ and d-diam$(V_n(x) \bigcap Y_0) < 1/n$ the sets $\bigcap_{n=1}^{\infty} U_n(x)$ and $\bigcap_{n=1}^{\infty} V_n(x)$ are one-point subsets of X_1 and Y_1 respectively. Hence $F(x) \subset \bigcap_{n=1}^{\infty} V_n(x)$ and since the latter set is a singleton and F is nonempty-valued, it follows that F is single-valued at the points of X_1.

Take further some $x_0 \in X_1$. Then $F(x_0) = \bigcap_{n=1}^{\infty} V_n(x_0)$. Let $F(x_0) \in V$ for some open V in Y. The sequence $\{V_n(x_0) \cap Y_0\}$ is a local base for the point $F(x_0)$ in Y_0. Since Y_0 is dense in Y and Y is regular it follows that $\{V_n(x_0)\}$ is a base for $F(x_0)$ in Y. Therefore $V_n(x_0) \subset V$ for some n. Now, for every $x \in U_n(x_0)$ we have $F(x) \subset V_n(x_0)$. Hence F is u.s.c. at x_0.

Finally, we prove that $F(X_1) = Y_1$ and F^{-1} is single-valued and u.s.c. at the points of Y_1. Take $y \in Y_1$. This y determines (in a unique way) sequences $\{U_n(y)\}$, $\{V_n(y)\}$ with the same properties as above. Now, by the construction $\bigcap_{n=1}^{\infty} U_n(y)$ is a one-point set in X_0, say x, and as seen from above $F(x) = \{y\}$. Hence $F|X_1: X_1 \to Y_1$ is onto and, again as above, F^{-1} is single-valued and u.s.c. at the points of Y_1. As a result, the restriction $F|X_1$ is a homeomorphism between X_1 and Y_1. □

Remark 5.5. It can be seen that Theorem 5.4 is true if $\mathrm{Dom}(F)$ is dense in X (rather than $\mathrm{Dom}(F) = X$) provided $\mathrm{Gr}(F)$ is closed in $X \times Y$ (see Theorem 3.3 and Remark 3.4).

6. Some other applications.

This section illustrates how the results obtained in the previous sections can be used in order to prove some known as well as new results. We start with two results concerning generic well-posedness of optimization problems.

(a) Generic well-posedness of optimization problems. Let X be a completely regular space and $C(X)$ denotes the space of all continuous real-valued and bounded functions on X endowed with the usual sup-norm $\|f\| = \sup\{|f(x)|: x \in X\}$. The couple (X, f) can be considered as a maximization problem:

find $x_0 \in X$ such that $f(x_0) = \sup\{f(x) \colon x \in X\} =: \sup(X, f)$. The maximization problem (X, f) is said to be *Tikhonov well-posed* [T] if it has unique solution and every maximizing sequence $\{x_n\}$ (that is $f(x_n) \to \sup(X, f)$) converges to this solution. (X, f) is *generalized Tikhonov well-posed* (see [FV] for the case of a metric space X and [ČKR1, ČKR2] for the general case of a topological space) if every maximizing net has a convergent subnet. Let $e(f)$, $f \in C(X)$, denote the unique continuous extension of f on βX, the Stone-Čech compactification of X. Let further, $M \colon C(X) \to \beta X$ be the multivalued mapping defined by $M(f) := \{x \in \beta X : e(f)(x) = \sup(X, f)\}$, $f \in C(X)$. It is well-known (see e.g. [ČK1, ČK2]) that M is open minimal and u.s.c.o. It is proved in [ČKR1, ČKR2, Propositions 1.10 and 1.12], that (X, f) is Tikhonov well-posed (resp. generalized Tikhonov well-posed) iff the set $M(f)$ is a singleton, necessarily lying in X (resp. $M(f) \subset X$).

Put $T := \{f \in C(X) \colon (X, f)$ is Tikhonov well-posed$\}$ and $GT := \{f \in C(X) \colon (X, f)$ is generalized Tikhonov well-posed$\}$. For compact X the following theorem was proved in [ČK1, Proposition 1 and Theorem 5], [ČK2, Proposition 1.2 and Theorem 2.3].

Theorem 6.1 [ČKR1, ČKR2, Theorem 3.5]. *The set T contains a dense G_δ-subset of $C(X)$ iff X contains a dense and completely metrizable subspace.*

Proof. Suppose that T contains a dense G_δ-subset T_1 of $C(X)$. Consider the mapping $M^{-1} \colon \beta X \to C(X)$. It is l.s.c. (since M is open) and has closed graph (since M is u.s.c.o.). Moreover, by Proposition 4.5(d), M^{-1} embraces every dense subset of $C(X)$, in particular T_1. Hence, by Theorem 4.7 there exist a dense G_δ-subset X_1 of βX and a continuous single-valued mapping $h \colon X_1 \to T_1$ which is a selection of M^{-1} on X_1. Obviously X_1 is Čech complete. Take $x \in X_1$. Then $h(x) \in T_1$ and hence $h(x) \in T$. Therefore $M(h(x)) \subset X$, showing that $x \in X$. Hence $X_1 \subset X$. On the other hand, since h is a selection of M^{-1} mapping X_1 into T_1 and M is u.s.c.o. and single-valued on T, it is seen that h is a homeomorphism between X_1 and $h(X_1) \subset T_1$. To finish, let us recall that every metrizable Čech complete space is completely metrizable.

Conversely, let X contain a dense and completely metrizable subspace X_1. Apply Theorem 5.3 to the mapping $M \colon C(X) \to \beta X$. Hence there is a dense G_δ-subset T_1 of $C(X)$ on which M is single-valued. Hence $T_1 \subset T$. □

Theorem 6.2 [ČKR1, ČKR2, Theorem 2.3]. *The set GT contains a dense G_δ-subset of $C(X)$ iff X is almost complete.*

Proof. Let GT contain a dense G_δ-subset T_1 of $C(X)$. As in the previous proof apply Theorem 4.6 to the mapping $M^{-1} \colon \beta X \to C(X)$. We get a dense G_δ-subset X_1 of βX (hence X_1 is Čech complete) such that $X_1 \subset X$.

Conversely, let X be almost complete. Then (see §4) X contains a dense and Čech complete subspace X_1. By Theorem 5.2 (applied to the mapping $M \colon C(X) \to \beta X$) there is a dense G_δ-subset T_1 of $C(X)$ such that M maps T_1 into X_1. Since $X_1 \subset X$ we get $T_1 \subset GT$. The proof is completed. □

Remark 6.3. Since for any $x \in \beta X$, $M^{-1}(x)$ is a convex closed subset of $C(X)$ and M^{-1} is l.s.c. in βX, the classical Michael selection theorem [M1] always gives a single-valued selection of $M^{-1}\colon \beta X \to C(X)$ which is defined on βX. The values of this selection, however, are not obliged to lie in T or GT.

(b) A generalized Lavrentieff theorem. The classical Lavrentieff theorem asserts that a homeomorphism f between A and B, where A and B are subsets of the complete metric spaces X and Y correspondingly, can be extended to a homeomorphism f' between some G_δ-subsets $A' \subset \mathrm{Cl}_X(A)$ and $B' \subset \mathrm{Cl}_Y(B)$, with $A \subset A'$ and $B \subset B'$. In [BLL] a class of spaces is characterized for which the Lavrentieff theorem is valid. This happens if every closed subset C of X (and D of Y) is Čech complete and has a G_δ-diagonal in $C \times C$ (in $D \times D$). We slightly change this setting of the problem and look for an extension f'' which is a homeomorphism between certain G_δ-subsets A'' of $\mathrm{Cl}_X(A)$ and B'' of $\mathrm{Cl}_Y(B)$ but these latter sets do not necessarilly contain A and B. Precisely, f'' is a homeomorphism between residual subsets of $\mathrm{Cl}_X(A)$ and $\mathrm{Cl}_Y(B)$ which contain A and B. In this case the corresponding Lavrentieff type theorem is true for a class of spaces which is strictly larger than that described in [BLL].

A completely regular topological space X is said to be from the class $\mathbf{M}$ if every closed subset of X contains a dense and completely metrizable subspace. The class $\mathbf{M}$ have been studied in [ČK1, ČK2, Š and ČKR1, ČKR2]. There exist Eberlein compacts (that is compact spaces which are homeomorphic to weak compact sets in a Banach space) which do not have a G_δ-diagonal. For example, take the one-point Aleksandroff compactification of the interval [0,1] where all points are open sets. By the criterion of Rosenthal [Ro] this space is an Eberlein compact, but its diagonal is not G_δ. On the other hand, Eberlein compacta are in the class $\mathbf{M}$ (see [Na]).

We now can state the following Lavrentieff type theorem:

Theorem 6.4. *Let X and Y be from the class* $\mathbf{M}$ *and f be a homeomorphism between $A \subset X$ and $B \subset Y$. Then f can be extended to a homeomorphism f' between residual subsets of* $\mathrm{Cl}_X(A)$ *and* $\mathrm{Cl}_Y(B)$ *containing A and B. Moreover, dense and completely metrizable subsets $A' \subset \mathrm{Cl}_X(A)$ and $B' \subset \mathrm{Cl}_Y(B)$ exist (not necessarily containing A and B) which are homeomorphic under f'.*

Proof. Take $\mathrm{Cl}_X(A)$ and $\mathrm{Cl}_Y(B)$ in X and Y respectively. Then $\mathrm{Cl}_X(A)$ and $\mathrm{Cl}_Y(B)$ contain dense and complete metrizable subspaces A_0 and B_0 respectively.

Define $F\colon \beta\mathrm{Cl}_X(A) \to \beta\mathrm{Cl}_Y(B)$ by putting $\mathrm{Gr}(F) = \mathrm{Cl}(\mathrm{Gr}(f))$. F and F^{-1} are nonempty-valued since $\mathrm{Dom}(f) = A$ and $\mathrm{Dom}(f^{-1}) = B$ are dense in the compact spaces $\beta\mathrm{Cl}_X(A)$ and $\beta\mathrm{Cl}_Y(B)$. Moreover, both F and F^{-1} are u.s.c.o. Further, since f is a homeomorphism between A an B it is easily checked that $\mathrm{Int}\,F^{\#}(V)$ and $\mathrm{Int}\,(F^{-1})^{\#}(U)$ are nonempty in $\beta\mathrm{Cl}_X(A)$ and $\beta\mathrm{Cl}_Y(B)$ provided U and V are nonempty open subsets in $\beta\mathrm{Cl}_X(A)$ and $\beta\mathrm{Cl}_Y(B)$ respectively. Hence, Theorem 5.4 gives the existence of dense G_δ-subsets A' of $\beta\mathrm{Cl}_X(A)$ and B' of $\beta\mathrm{Cl}_Y(B)$ such that $F|A'$ is a homeomorphism between A' and B'. Observe that (again by Theorem 5.4) $A' \subset A_0$ and $B' \subset B_0$, hence A' and B' are also completely metrizable.

Since f is a homeomorphism the restrictions of F on A and of F^{-1} on B coincide with f and f^{-1} respectively. Put $A'' := A \cup A'$ and $B'' := B \cup B'$. A'' and B'' are residual subsets of $\mathrm{Cl}_X(A)$ and $\mathrm{Cl}_Y(B)$ containing A and

B. To finish, remember that F and F^{-1} are u.s.c.o. Therefore $F|A''$ is a homeomorphism between A'' and B''. □

(c) Generic differentiability of convex functions. Let $(E, \|\cdot\|)$ designate a real Banach space with norm $\|\cdot\|$. By E^*, as usual, we denote the dual of E. That is, E^* is the space of all continuous linear functionals in E endowed with the dual norm $\|x^*\| = \sup\{|\langle x, x^*\rangle| : x \in B\}$, $x^* \in E^*$, where $B := \{x \in E : \|x\| \leq 1\}$ is the closed unit ball in E and $\langle\cdot, \cdot\rangle$ designates the usual duality between E and E^*. The weak star topology in E^* will be denoted by w^*.

Let $f : E \to \mathbf{R}$ be a continuous convex function. The *subdifferential* ∂_f of f, is a multivalued mapping acting from E into E^*, defined by the formula: $\partial_f(x) := \{x^* \in E^* : \langle y - x, x^*\rangle \leq f(y) - f(x)$ for every $y \in E\}$, $x \in E$. It easily follows by this definition that ∂_f is a monotone mapping, i.e., for every $x_1, x_2 \in E$ and for every $x_1^* \in \partial_f(x_1)$ and $x_2^* \in \partial_f(x_2)$ one has $\langle x_1 - x_2, x_1^* - x_2^*\rangle \geq 0$.

The following facts are well-known and may be found in [Ph].

(i) $\partial_f : E \to E^*$ is a norm-to-w^* u.s.c.o. mapping with $\mathrm{Dom}(\partial_f) = E$;
(ii) f is Gâteaux differentiable at $x_0 \in E$ iff $\partial_f(x_0)$ is a singleton;
(iii) f is Fréchet differentiable at $x_0 \in E$ iff ∂_f is single-valued and norm-to-norm u.s.c. at x_0.

In 1933 Mazur [Ma] proved that if E is a separable Banach space, then f is Gâteaux differentiable at the points of some residual subset of E (in such a case we say that f is generically Gâteaux differentiable). In 1968 Asplund [As] exhibited very general sufficient conditions under which f is generically Gâteaux or Fréchet differentiable.

Let $d(\cdot, \cdot)$ be a metric in E^*. It is said that d *fragments* E^* (see [JR]) if for every bounded subset A of E^* and every $\varepsilon > 0$ there exists some w^*-open subset V of E^* such that $V \cap A \neq \varnothing$ and $d\text{-diam}(V \cap A) < \varepsilon$. It is well-known that E^* has Radon-Nikodym Property (RNP) iff E^* is fragmented by the metric generated by the dual norm in E^*. Another important example is provided in the paper of Ribarska [Ri]: if the norm in E is Gâteaux differentiable at each $x \in E$, $x \neq 0$, then E is fragmented by some metric d.

The following theorem is well-known.

Theorem 6.5. *Let E be a Banach space with dual E^* fragmented by some metric d and $f : E \to \mathbf{R}$ be a continuous convex function. Then there exists a dense G_δ-subset D of E at the points of which f is Gâteaux differentiable. If the metric d is generated by the dual norm in E^* then at the points of D f is Fréchet differentiable.*

Proof. Since $\partial_f : (E, \|\cdot\|) \to (E^*, w^*)$ is u.s.c.o. there exists some minimal u.s.c.o. $F : (E, \|\cdot\|) \to (E^*, w^*)$ which is a selection of ∂_f. The following lemma gives us the possibility to apply Theorem 5.1.

Lemma 6.6. *The mapping $F : (E, \|\cdot\|) \to (E^*, d)$ is fragmented by the metric d.*

Proof. Consider the closed unit ball B^* in E^*. Since F is norm-to-w^* u.s.c.o. the sets $F^{-1}(nB^*)$ are closed in X for every $n \geq 1$. Take an arbitrary nonempty open set U of E and positive ε. Observe that $U \subset \bigcup_{n=1}^{\infty} F^{-1}(nB^*)$. By the Baire theorem $U_1 := U \cap \mathrm{Int}F^{-1}(nB^*) \neq \varnothing$ for some $n \geq 1$. Since F is minimal we get $F(U_1) \subset nB^*$ (Proposition 2.1 (b)). On the other hand, d fragments E^*, hence there exists a w^*-open set V of E^* such that $V \cap F(U_1) \neq \varnothing$ and d-diam$(V \cap F(U_1)) < \varepsilon$. Again by the minimality of F (Proposition 2.1 (c)) one gets a nonempty open $U' \subset U_1$ such that $F(U') \subset V$. Therefore d-diam$(F(U')) < d$-diam$(V \cap F(U_1)) < \varepsilon$. The proof of the lemma is completed. □

Now, let us go back to the proof of Theorem 6.5. By Theorem 5.1 there exists a dense G_δ-subset D of E at the points of which F is single-valued and norm-to-d u.s.c. We will show further that

(i) $\partial_f(x) = F(x)$ for every $x \in D$ (i.e., F is Gâteaux differentiable at the points of D);

(ii) if the metric d is generated by the dual norm in E^* then ∂_f is single-valued and norm-to-norm u.s.c. at any $x \in D$ (this would imply that f is Fréchet differentiable at the points of D).

We will prove (ii) now. Let $x_0 \in D$ and $\varepsilon > 0$. Then $F(x_0) = \{x_0^*\}$ for some $x_0^* \in E^*$ and $x_0^* \in \partial_f(x_0)$. Since F is norm-to-norm u.s.c. at x_0 there exists some open U of X such that $x_0 \in U$ and $F(U) \subset x_0^* + \varepsilon B^*$. It suffices to show that $\partial_f(U) \subset x_0^* + \varepsilon B^*$. Suppose this is not the case and take some $x_1^* \in \partial_f(x_1) \backslash \{x_0^* + \varepsilon B^*\}$ where $x_1 \in U$. Then there exists $h \in E$, $\| h \| = 1$, which strongly separates x_1^* from the close convex ball $x_0^* + \varepsilon B^*$, i.e., for some $\delta > 0$ the w^*-open set $H_\delta :\, = \{x^* \in E^* : \langle h, x^* \rangle > \langle h, x_1^* \rangle - \delta\}$ does not intersect $x_0^* + \varepsilon B^*$.

Consider, for $t > 0$, the point $x(t) :\, = x_1 + th$. By the monotonicity of ∂_f we have

$$0 \leq \langle x(t) - x_1, x^* - x_1^* \rangle = t \langle h, x^* - x_1^* \rangle \quad \text{for every } x^* \in \partial_f(x(t)).$$

This means that $\partial_f(x(t)) \subset H_\delta$ for every $t > 0$. However, when t is small enough $x(t) \in U$ and hence $F(x(t)) \subset x_0^* + \epsilon B^*$. This is a contradiction since F is a selection of ∂_f. The proof of (ii) is completed. The proof of (i) is even simpler. □

In the next theorem the Banach space E is identified with its natural embedding in its second dual E^{**}.

Theorem 6.7. *Let E be a separable Banach space and $f: E^* \to \mathbf{R}$ be a continuous convex function such that the set $A := \{x^* \in E^* : \partial_f(x^*) \cap E \neq \varnothing\}$ is residual in the norm topology in E^*. Then f is Fréchet differentiable on a dense G_δ-subset A' of E^* and $\partial_f(x^*) \in E$ for every $x^* \in A'$.*

Proof. Consider the mapping $F: A \to E$ defined by $F(x^*) := \partial_f(x^*) \cap E$, $x^* \in A$. Evidently $\mathrm{Dom}(F) = A$. On the other hand, since ∂_f has a closed graph, the mapping F has a closed graph too. Hence, by Theorem 4.16 there exist a residual subset A' of A and a continuous single-valued mapping h : $A' \to E$ which is a selection of F. Hence A' is residual in E^*. Arguments similar to the used in the proof of (ii) from Theorem 6.5 show that for every $x^* \in A'$ $\partial_f(x^*) = h(x^*)$ and ∂_f is norm-to-norm u.s.c. at x^*. This means that f is Fréchet differentiable at the points of A'. □

Put, further, $BP := \{x^* \in E^* : x^*$ attains its maximum on the unit ball $B\}$. According to the famous Bishop-Phelps theorem [BPh], the set BP is dense in the norm topology in E^*.

Corollary 6.8. *Let E be a separable Banach space such that the Bishop-Phelps set BP is residual in E^*. Then the norm in E^* is Fréchet differentiable at the points of some residual subset A' of E^* and for every $x^* \in A'$ $\partial_{\|\cdot\|}(x^*) \in B$. In particular, for every $x^* \in A'$ the maximization problem (B, x^*) is Tikhonov well-posed.*

Proof. Let B^{**} be the closed unit ball in E^{**}. It is well known that $\partial_{\|\cdot\|}(x^*) = \{x^{**} \in B^{**} : \langle x^*, x^{**}\rangle = \|x^*\|\}$ and that $\partial_{\|\cdot\|}(x^*) \cap E = \partial_{\|\cdot\|}(x^*) \cap B \neq \varnothing$ iff x^* attains its maximum on B. Therefore, $BP = \{x^* \in E^* : \partial_{\|\cdot\|}(x^*) \cap E \neq \varnothing\}$. It remains to apply Theorem 6.7. □

Remark 6.9. As shown in [KG, Theorem 3.5] the last two statements are valid for all Banach spaces which admit an equivalent locally uniformly rotund norm.

(d) Metric projections and antiprojections in Banach spaces. Let again E be a real Banach space with norm $\|\cdot\|$. Denote by S the unit sphere of E, i.e., the set $\{x \in E : \|x\| = 1\}$. Recall that the norm $\|\cdot\|$ in E is *locally uniformly rotund* if for every $x_0, x_n \in S$ such that $(1/2)\|x_0 + x_n\| \to 1$, it follows that $x_n \to x_0$. $\|\cdot\|$ is *strictly convex* if S does not contain line segments.

Let $A \subset E$ be a nonempty subset of E. The *metric projection* generated by A is the multivalued mapping $P_A : E \to A$ defined by $P_A(x) : = \{y \in A : \|x - y\| = \inf\{\|x - y'\| : y' \in A\}\}$. The conjecture that the set $\{x \in E : P_A(x) = \varnothing$ or $P_A(x)$ is a singleton$\}$ is residual in E provided the norm in E is strongly convex was stated by Stečkin [St]. Up to now there are partial positive answers to this question (see [FaZh, Ko1, Ko2, L, St, Za, Zh]).

Analogously, if A is a nonempty and bounded subset of E one can consider the *metric antiprojection* mapping $Q_A : E \to A$ defined by $Q_A(x) : = \{y \in A : \|x - y\| = \sup\{\|x - y'\| : y' \in A\}\}$. From the above point of view the metric antiprojection is investigated in [PaKa, Zh].

We prove here the original result of [St] about metric projections in a slightly stronger form containing Theorem 1.8 from [Zh].

Theorem 6.10. *Let E have a locally uniformly rotund norm and A be closed (resp. closed and bounded). Then there exists a dense G_δ-subset E_1 of E at the points of which P_A (resp. Q_A) is no more than single-valued and u.s.c. Moreover, if* $\mathrm{Dom}(P_A)$ *(resp.* $\mathrm{Dom}(Q_A)$*) is dense in E then $E_1 \subset$* $\mathrm{Dom}(P_A)$ *(resp. $E_1 \subset$* $\mathrm{Dom}(Q_A)$*).*

Proof. We prove the theorem only for metric projections. The case of antiprojections is analogous.

Let $X := \mathrm{Cl}\,\mathrm{Dom}(P_A)$. The set $E \backslash X$ is open and if it is nonempty then obviously for every $x \in E \backslash X$ $P_A(x) = \varnothing$ and P_A is u.s.c. at x. So consider $Y := \mathrm{Int}\,\mathrm{Cl}\,\mathrm{Dom}(F)$. The set $X \backslash Y$ is nowhere dense in E. Hence, if $Y = \varnothing$, we are done. On the other hand, if $Y \neq \varnothing$ and one proves that there is a dense G_δ-subset X_1 of X such that P_A is single-valued and u.s.c. in X at each point of X_1, then one can easily get the conclusion of the theorem.

So, consider P_A only in X. Observe that X is a complete metric space, hence a Baire space, and moreover $\mathrm{Dom}(P_A)$ is dense in X. On the other hand, by the fact that A is closed one can easily see that $\mathrm{Gr}(P_A)$ is a closed subset of $X \times A$. Further, let $x_0 \in X$ and $P_A(x_0) \neq \varnothing$. By Lemma 1.7 from [Zh] for every $y \in P_A(x_0)$ the metric projection P_A is upper semicontinuous and single-valued at each point x' from the interior of the line segment $[x_0, y]$. Since for every such x' $P_A(x') = y$ we easily get that $\mathrm{Int}\, P_A^{\#}(V)$ is dense in $P_A^{-1}(V)$ for every V open in A. Hence, by Theorem 5.3, there exists a dense G_δ-subset X_1 of X such that $X_1 \subset \mathrm{Dom}(P_A)$ and P_A is single-valued and u.s.c. at the points of X_1. The proof is completed. □

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